Time Optimal Quantum State Control:

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Synopsis:

There exists a Hamiltonian operator, for a given constraint, that drives the system from an initial state to a target final state in least time. This Hamiltonian operator is a time-dependent matrix with symmetry properties that represent the internal spectroscopy of the atomic system.

It is possible to find the time optimal trajectory for the quantum state once the Hamiltonian is defined, and hence define a minimum time required for the evolution in question.

Results indicate that the time-dependence contained within the Hamiltonian matrix for particular choices of groups on SU(N) is of a periodic nature. This enables great simplification in analysis as many results from the theory of periodic differential equations may be directly applied to physical problems.

In the practical area of semiconductor design and quantum dots on substrates, this requires that the coupling between the dots be of an AC-type. Quantum dot scenarios are ideal for the application of this method as the positions in space which the particle may occupy are limited to a finite set of locations.

Mathematical apparatus:

Matrices:

$$\tilde{A} = [A_{ij}] \;\; 1 \leqslant i, j \leqslant n$$

$$\mathrm{Tr}(\tilde{A}) = \sum_k A_{kk}$$

State vector: $|\psi> = [c_1(t), c_2(t), c_3(t), ..., c_n(t)]^T$

Hamiltonian matrix:

$$\tilde{H} = \begin{pmatrix} H_{11}(t) & ... & H_{1n}(t) \\ ... & & ... \\ H_{n1}(t) & ... & H_{nn}(t) \end{pmatrix} \; ; \; H_{jk}(t) = H^*_{kj}(t) \; ; \; \sum_k H_{kk}(t) = 0$$

Constraint matrix:

$$\mathrm{Tr}(\tilde{H}\hat{F}) = 0 \quad ; \quad \mathrm{Tr}(\hat{F}) = 0 \quad ; \quad \hat{F}^+ = \hat{F}$$

Quantum Brachistochrone Equation:

$$i\frac{d\hat{A}(t)}{dt} = \hat{H}(t)\hat{A}(t) - \hat{A}(t)\hat{H}(t) \qquad \text{for any Hermitean } \hat{A}$$

Take the ansatz $\hat{A} = \tilde{H}(t) + \hat{F}(t)$, and using $[\tilde{H}(t), \tilde{H}(t)] = 0$ we may rewrite the Heisenberg equation of motion as:

$$i\frac{d}{dt}(\tilde{H}(t) + \hat{F}(t)) = \tilde{H}(t)\hat{F}(t) - \hat{F}(t)\tilde{H}(t)$$

with our quantum state obeying the Schrodinger equation:

$$i\frac{d|\psi(t)>}{dt} = \tilde{H}(t)|\psi(t)>$$

The time dependent Hamiltonian has the property that it may not commute at different times, i.e. $[\tilde{H}(t_1), \tilde{H}(t_2)] \neq 0$ for $t_1 \neq t_2$.

Some simple time optimal operators and boundary conditions:

SU(2):

$$\tilde{H} = |\varepsilon(0)| \begin{pmatrix} 0 & \exp(-2i\Omega t) \\ \exp(+2i\Omega t) & 0 \end{pmatrix} \;;\; \hat{F} = \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix} \quad w = \text{const.}$$

$$|\psi(0)> = \frac{1}{\sqrt{2}}[1,1]^T \;;\; |\psi(t_f)> = \frac{1}{\sqrt{2}}[1,-1]^T$$

$$|\varepsilon(0)| \times t_f = \frac{\pi}{2}$$

SU(3):

$$\tilde{H} = |\varepsilon(0)| \begin{pmatrix} 0 & \cos(kt) & 0 \\ \cos(kt) & 0 & e^{-i\theta}\sin(kt) \\ 0 & e^{+i\theta}\sin(kt) & 0 \end{pmatrix} \;;\; \hat{F} = \begin{pmatrix} \omega_1 & 0 & \kappa \\ 0 & \omega_2 & 0 \\ \kappa^* & 0 & \omega_3 \end{pmatrix} = \text{const.}$$

$$|\psi(0)> = [1,0,0]^T \;;\; |\psi(t_f)> = [0,0,1]^T$$

$$|\varepsilon(0)| \times t_f = \frac{\sqrt{3}\pi}{2}$$

SU(4):

$$\tilde{H} = |\varepsilon(0)|(\hat{\sigma}_x \otimes \hat{\sigma}_x - \hat{\sigma}_y \otimes \hat{\sigma}_y) \;;\; \text{Tr}(\tilde{H}\hat{F}) = 0$$

$$|\psi(0)> = [1,0,0,0]^T \;;\; |\psi(t_f)> = \frac{1}{\sqrt{2}}[1,0,0,-i]^T$$

$$|\varepsilon(0)| \times t_f = \frac{\pi}{8}$$

In general it seems that, given the existence of a Hamiltonian operator with sufficient smoothness, the Heisenberg Uncertainty Principle for energy and time is obeyed in the form:

$$(\text{Energy strength of Hamiltonian}) \times (\text{time of evolution for state}) = \text{constant}$$

Also, the number of possible choices of a Hamiltonian and constraint law for SU(N) is equal to:

$$\#(N) = 2^{N^2-1}$$

which diverges rather rapidly as $N \to \infty$. It is worse than the proverbial needle in a haystack!